PRMO 2018

Q	1	2	3	4	5	6	7	8	9	10
Key	17	8	70	12	84	18	14	80	81	24
Q	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Key	29	88	24	19	21	55	30	16	33	17
Q	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
Key	63	6	14	27	81	62	90	24	30	34

OLYMPIAD: MATHEMATICS 2018

1. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number of pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n?

   Solution

   Pages are numbered from 1 to p : 1^{st} volume

   Pages are numbered from p + 1 to q : 2^{nd} volume

   Pages are numbered from q + 1 to n : 3^{rd} volume

   Given $q - p = 50 + p$

   $$n - q = \frac{3}{2}(q - p)$$

   $1 + (p + 1) + (q + 1) = 1709$

   $2p = q - 50 \Rightarrow p + 50 + 2p = 1706$

   $\Rightarrow 3p = 1656$

   $\Rightarrow p = 552$

   $\Rightarrow q = 1152$

   $$\therefore\ n - 1152 = \frac{3}{2}(1152 - 552)$$

   $\Rightarrow n = 2057$

   $\Rightarrow$ Largest prime factor of $2057 = 11 \times 11 \times 17$ is 17

2. In a quadrilateral ABCD, it is given that AB = AD = 13, BC = CD = 20, BD = 24 If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r?

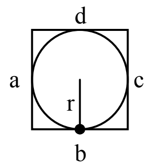
   Solution

   Semiperimeter (s) $= \frac{a+b+c+d}{2} = \frac{20+13+20+13}{2} = \frac{66}{2} = 33$

   Area of quadrilateral (k) $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$

   $= \sqrt{(33-20)(33-13)(33-20)(33-13)}$

   $= 20 \times 13 = 260$



   $$r = \frac{k}{s} = \frac{260}{33} = 7.87$$

   Integer closes to r = 8

3. Consider all 6-digit numbers of the form abccba where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7.

   Solution

   abccba (b is odd)

   $= 100000a + 10000b + 1000c + 100c + 10b + 1 \cdot a$

   $= 10^5 a + 10^4 b + 10^3 c + 10^2 c + 10^1 \cdot b + 1 \cdot a$

   $= a(10^5 + 1) + (10^4 + 10)b + (10^3 + 10^2)c$

   $= a(10^3)(10^2) + a + 10b(1001) + (100)(11)c$

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$= a(1001 - 1)100 + a + 10b\,(1001) + (100)\,(11)c$
$= a(1001) - 100a + a + 10b\,(1001) + (100)\,(11)c$
$= a(7\cdot11\cdot13\cdot100) - 99a + 10b\,(7\cdot11\cdot13) + (98 + 2)\,(11)\,c$
$= a(7\cdot11\cdot13\cdot100) - 98a - a + 10b(7\cdot11\cdot13) + 98\cdot11\cdot c + 2\cdot11\cdot c$
$= a(7\cdot11\cdot13\cdot100) - 7.14a - a + 10b\,(7\cdot11\cdot13) + 7\cdot14\cdot11\cdot c + 21c + c$
$= 7p + (c - a)$ where p is an integer
Now if $c - a$ is a multiple of 7
$c - a = 7, 0, -7$
Hence number of ordered pairs of (a, c) is 14
Since b is odd
Number of such number $= 14 \times 5 = 70$

4. The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is, 1, 6, 5, 8, 9, 0 are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation.

Solution

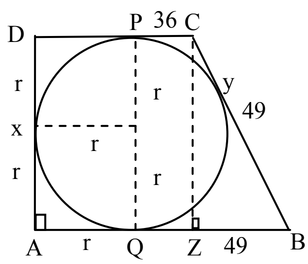
Given $166 \times 56 = 8590$ is invalid for some base $b \geq 10$

$(b^2 + 6b + 6)\,(5b + 6) = (8b^3 + 5b^2 + 9b + 0)$
$\Rightarrow\ 5b^3 + 36b^2 + 66b + 36 = 8b^3 + 5b^2 + 9b$
$\Rightarrow\ 3b^3 - 31b^2 - 57b - 36 = 0$
$\Rightarrow\ b = 12$

5. Let ABCD be a trapezium in which AB || CD and AD ⊥ AB. Suppose ABCD has an incircle which touches AB at Q and CD at P. Given that PC = 36 and QB = 49, find PQ.

Solution

In incircle touch BC at Y
CP = CY = 36, BY = 49 = BQ
AQ = PD = r, AD = 2r
$\overline{CZ} \perp \overline{QB}$



$CZ^2 = 85^2 - (49 - 36)^2$
$= 85^2 - 13^2$
$= 98 \times 72$
$= 49 \times 144$
$CZ = 7 \times 12$
$CZ = 84$
$PQ = 84$

6. Integers a, b, c satisfy $a + b - c = 1$ and $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$?

Solution

Here the given equations are
$a + b - c = 1$ and $a^2 + b^2 - c^2 = -1$
$\Rightarrow c = a + b - 1$
Putting the value of $c = a + b - 1$ in $a^2 + b^2 - c^2 = -1$
$a^2 + b^2 - (a + b - 1)^2 = -1$

$\Rightarrow\ a^2 + b^2 - (a^2 + b^2 + 1 + 2ab - 2b - 2a) = -1$

$\Rightarrow\ a^2 + b^2 - a^2 - b^2 - 1 - 2ab + 2b + 2a = -1$

$\Rightarrow\ 2a + 2b - 2ab = 0$

$\Rightarrow\ a + b = ab$

$\Rightarrow\ b = \dfrac{a}{a-1}$

Since a, b are integers,

So, the possible values of a and b are

Possibility 1 : $a = 0 \Rightarrow b = 0$ and $c = -1$

Which also satisfy $a^2 + b^2 - c^2 = -1$

Then $a^2 + b^2 + c^2 = +1$

Possibility 2: $a = 2 \Rightarrow b = 2$ and $c = 3$

Which also satisfy $a^2 + b^2 - c^2 = -1$

Then $a^2 + b^2 + c^2$

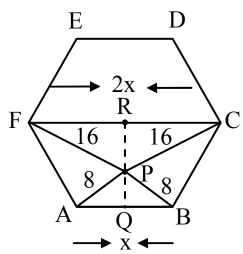
$= 4 + 4 + 9$

$= 17$

The sum of all possible values is 17 + 1 = 18

7. A point P in the interior of a regular hexagon is at distances 8, 8, 16 units from three consecutive vertices of the hexagon, respectively. If r is radius of the circumscribed circle of the hexagon. What is the integer closest to r?

Solution



ABCDEF in a regular hexagon

CF = 2AB

2PA = PC

2PB = PF

$\Delta PAB \sim \Delta PFC$

$\overline{PR} \perp \overline{FC}$, $\overline{PQ} \perp \overline{AB}$ (R is incentre of hexagon)

FC = 2AB = 2x

$\frac{1}{3} \times \frac{\sqrt{3}x}{2} = \sqrt{64 - \frac{x^2}{4}} \Rightarrow \frac{x^2}{12} = 64 - \frac{x^2}{4}$

$\frac{4x^2}{12} = 64$

$x = 8\sqrt{3}$

$r = 8 \times 1.732$

$= 13.856$

$= 14$ (Nearest integer)

8. Let AB be a chord of a circle with centre O. Let C be a point on the circle such that $\angle ABC = 30^\circ$ and O lies inside the triangle ABC. Let D be a point on AB such that $\angle DCO = \angle OCB = 20^\circ$. Find the measure of $\angle CDO$ in degrees.

Solution

$\angle DCO = \angle OCB = 20^\circ$

$\angle OBC = 20^\circ$

$\angle OBA = 10^\circ$

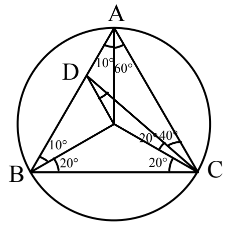
$\angle OAB = 10^\circ$

$\angle BOC = 140^\circ$

$\angle A = 70^\circ$

$\angle OAC = 60^\circ$

$\angle ACD = 40^\circ$



C is circumcentre of ΔAOD

$\angle OCD = 2\angle OAD$

$\angle AOD = 2\angle OAD = 20^\circ$

$\angle DOC = \angle AOD + \angle AOC$

$= 20 + 60$

$= 80^\circ$

$\angle ODC = 180^\circ - (\angle DOC + \angle OCD)$

$= 180^\circ - (80 + 20)$

$= 80^\circ$

9. Suppose a, b are integers and a + b is a root of $x^2 + ax + b = 0$. What is the maximum possible value of b^2 ?

Solution

Since a + b is a root of $x^2 + ax + b = 0$

$(a + b)^2 + a(a + b) + b = 0$

$\Rightarrow 2a^2 + (3b)a + b(1 + b) = 0$

$$a = \frac{-3b \pm \sqrt{9b^2 - 4(2)(b(1+b))}}{4}$$

$$= \frac{-3b \pm \sqrt{b^2 - 8b}}{4}$$

Since $a, b \in Z$, $b^2 - 8b$ is a perfect square

Also $a^2 - 4b \geq 0$

$\Rightarrow$ Maximum value of b is 9

$\Rightarrow$ Maximum value of b^2 is 81

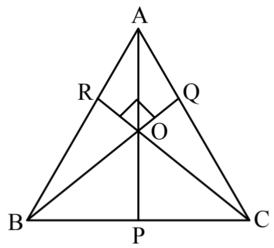
10. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $\dfrac{(BC^2 + CA^2 + AB^2)}{100}$

Solution

Here O divides AP in the ratio 2 : 1 and given AP = 30

$\therefore$ OA = 20 and OP = 10

$\therefore$ OA = OB = OC = 20



Now, $\dfrac{BC^2 + CA^2 + AB^2}{100}$

$$= \frac{(OB^2 + OC^2) + (OA^2 + OC^2) + (OA^2 + OB^2)}{100}$$

$$= \frac{2(OA^2 + OB^2 + OC^2)}{100}$$

$$= \frac{2 \times [20^2 + 20^2 + 20^2]}{100}$$

$$= \frac{2 \times [3 \times (20)^2]}{100}$$

$$= \frac{2 \times 3 \times 400}{100} = 2 \times 3 \times 4 = 24$$

11. There are several tea cups in the kitchen, some with handles and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?

Solution

If p is the number of cups without handle

q is the number of cups with handle, then

Given ${}^pC_2\ {}^qC_3 = 1200$

$\Rightarrow \frac{p(p-1)}{2!} \cdot \frac{q(q-1)(q-2)}{3!} = 1200$

$\Rightarrow$ $p(p-1)\, q(q-1)\, (q-2) = 2^6 \cdot 3^2 \cdot 5^2$

$\Rightarrow$ $25(24)\, 4(3)(2) = 2^6 \cdot 3^2 \cdot 5^2$

$\therefore$ $p = 25, q = 4$

$\Rightarrow$ $p + q = 29$

12. Determine the number of 8-tuples $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_8)$ such that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_8 \in \{1, -1\}$ and $\varepsilon_1 + 2\varepsilon_2 + 3\varepsilon_3 + \ldots + 8\varepsilon_8$ is a multiple of 3

Solution

$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_8 \in \{1, -1\}$ and $\varepsilon_1 + 2\varepsilon_2 + 3\varepsilon_3 + \ldots + 8\varepsilon_8$

$\Rightarrow$ $(\varepsilon_1 + 4\varepsilon_4 + 7\varepsilon_7) + (2\varepsilon_2 + 5\varepsilon_5 + 8\varepsilon_8) + (\varepsilon_3 + 2\varepsilon_6)$ … (1)

$\Rightarrow$ $(\varepsilon_1 + \varepsilon_4 + \varepsilon_7) + 2(\varepsilon_2 + \varepsilon_5 + \varepsilon_8) + 3p$ … (2)

where p is an integer

$\Rightarrow$ $(\varepsilon_1 - \varepsilon_2) + (\varepsilon_4 - \varepsilon_5) + (\varepsilon_7 - \varepsilon_8) + 3q$ … (3)

where q is an integer

$\varepsilon_1 - \varepsilon_2 = 2, 0, -2$ similarly we can write others

Let $\varepsilon_1 - \varepsilon_2 = k_{12}$ and $\varepsilon_4 - \varepsilon_5 = k_{45}$ and $\varepsilon_7 - \varepsilon_8 = k_{78}$

Now put the value in equation (3)

$k_{12} + k_{45} + k_{78} + 3q$

$k_{12} = k_{45} = k_{78} = 0$ (Total 8 cases)

$k_{12} = k_{45} = k_{78} = (2 \text{ or } -2)$ (Total 2 cases)

$k_{12} = 2, k_{45} = -2, k_{78} = 0 \Rightarrow (2 \times 6)$ cases

So number of tuples = $22 \times 4 = 88$

13. In a triangle ABC, right-angled at A, the altitude through A and the internal bisector of $\angle A$ have lengths 3 and 4, respectively. Find the length of the median through A.

Solution

In ΔABC, $\angle A = 90°$

$AX = 3, AY = 4$

Let $AZ = m$

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By angle bisector theorem

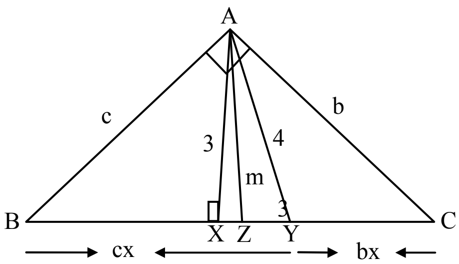
$$\frac{BY}{YC}=\frac{AB}{AC}=\frac{c}{b}$$

$$\frac{1}{2}bc=\frac{1}{2}(bx+cx)\times 3$$

$$x=\frac{bc}{3(b+c)} \qquad \dots (1)$$

$$\frac{1}{2}bc=\frac{1}{2}4c\frac{1}{\sqrt{2}}+\frac{1}{2}4b\frac{1}{\sqrt{2}}$$

$$bc=2\sqrt{2}(b+c) \qquad \dots (2)$$



From eqns. (1) and (2)

$$x=\frac{2\sqrt{2}}{3}$$

$$b^2 + c^2 = (bx + cx)^2$$

$$b^2+c^2=\frac{8}{9}(b^2+c^2+2bc)$$

$$b^2 + c^2 = 16bc \qquad \dots (3)$$

$$b^2 + c^2 = (bx + cx)^2$$

$$b^2+c^2=\frac{b^2c^2}{9(b+c)^2}\times(b+c)^2$$

$$9b^2 + 9c^2 = b^2c^2$$

$$9(16bc) = b^2c^2 \qquad (\because\ (3))$$

$$bc = 9 \times 16 \qquad \dots (4)$$

By Appollonius Theorem

$$2\left[m^2+\left(\frac{cx+bx}{2}\right)^2\right]=b^2+c^2$$

$$2m^2+\frac{b^2c^2}{9(b+c)^2}\times\frac{(c+b)^2}{2}=b^2+c^2$$

$$2m^2+\frac{b^2c^2}{18}=b^2+c^2$$

$$36m^2 = b^2c^2$$

$$6m = bc$$

$$m=\frac{bc}{6}=\frac{9\times 16}{6}=24$$

14. If x = cos 1° cos 2° cos 3° … cos 89° and y = cos 2° cos 6° cos 10° … cos 86°, then what is the integer nearest to $\frac{2}{7}\log_2\left(\frac{y}{x}\right)$?

Solution

x = cos 1° cos 2° cos 3° … cos 89° and

y = cos 2° cos 6° cos 10° … cos 86°

$$\frac{y}{x}=\frac{\cos 2°\cos 6°\ \cos 10°....\ \cos 86°}{\cos 1°\ \cos 2°\cos 3°...\ \cos 89°}$$

$$=2^{44}\times\sqrt{2}\cdot\frac{\cos°\cdot\cos 6°\cdot\cos 10°....\cos 86°}{\sin 2°\cdot\sin 4°\cdot\sin 6°....\cos 88°}$$

$$= 2^{\frac{89}{2}} \cdot \frac{\sin 4° \cdot \sin 8° \cdot \sin 12° .... \sin 88°}{\sin 2° \cdot \sin 4° \cdot \sin 6° ... \sin 88°}$$

$$= \frac{2^{\frac{89}{2}}}{\cos 4° \cdot \cos 8° \cdot \cos 12° ... \cos 88°}$$

$$= \frac{2^{\frac{89}{2}}}{\left(\frac{1}{2^{22}}\right)} = 2^{\frac{89}{2}} + 22 = 2^{\frac{133}{2}}$$

$$\frac{2}{7} \cdot \log_2\left(\frac{y}{x}\right) = \frac{2}{7} \cdot \log_2 \cdot 2^{\frac{133}{2}} = \frac{2}{7} \times \frac{133}{2} = 19$$

15. Let a and b be natural numbers such that $2a - b$, $a - 2b$ and $a + b$ are all distinct squares. What is the smallest possible value of b?

Solution

Let $2a - b = (p_1)^2$ … (1)

$a - 2b = (p_2)^2$ … (2)

$a + b = (p_3)^2$ … (3)

where p_1, p_2 and p_3 are distinct

now adding (2) and (3) then we will get

$2a - b = (p_2)^2 + (p_3)^2$ $\because$ $2a - b = (p_1)^2$

$(p_1)^2 = (p_2)^2 + (p_3)^2$

If b is least then $(p_3)^2$ and $(p_2)^2$ should be least and also multiple by 3

So, $(p_2)^2 = a - 2b = 9^2$ and $(p_3)^2 = a + b = (12)^2$

$\Rightarrow$ $(p_3)^2 - (p_2)^2 = 3b$

$(12)^2 - (9)^2 = 3b$

$144 - 81 = 3b$

$63 = 3b$

$b = 21$ (least value)

16. What is the value of $\sum\limits_{\substack{1 \le i < j \le 10 \\ i+j=\text{odd}}} (i + j) - \sum\limits_{\substack{1 \le i < j \le 10 \\ i+j=\text{even}}} (i + j)$?

Solution

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{1,10} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{2,10} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} & a_{3,10} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} & a_{4,10} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} & a_{5,10} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} & a_{6,10} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} & a_{7,10} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} & a_{8,10} \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} & a_{9,10} \\ a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} & a_{10,10} \end{bmatrix}_{10 \times 10} \quad i < j$$

For $i < j$

For $(i + j = \text{odd})$

$$\sum_{1\le i<j\le 10}(i+j) = (1+2+1+4+1+6+1+8+1+10)+(2+3+2+5+2+7+2+9)+(3+4+3+6+3+8+3+10)$$
$$+(4+5+4+7+4+9)+(5+6+5+8+5+10)+(6+7+6+9)$$
$$+(7+8+7+10)+(8+9)+(9+10)$$
$$=275$$

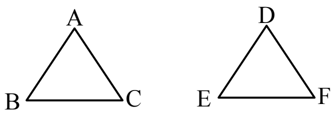
For (i + j = even)

$$\sum_{1\le i<j\le 10}(i+j) = (1+3+1+5+1+7+1+9)+(2+4+2+6+2+8+2+10)+(3+5+3+7+3+9)$$
$$+(4+6+4+8+4+10)+(5+7+5+9)+(6+8+6+10)+(7+9)+(8+10)$$
$$=220$$

Then $\sum_{\substack{10\le i<j\le 10\\ i+j=odd}}(i+j)-\sum_{\substack{10\le i<j\le 10\\ i+j=even}}(i+j) = 275-220=55$

17. Triangles ABC and DEF are such that $\angle A = \angle D$, AB = DE = 17, BC = EF = 10 and AC – DF = 12. What is AC + DF?

Solution



Let DF = x, AC = 12 + x

Since $\angle A = \angle D$

Using cosine rule, cos A = cos D

$$\frac{17^2+(12+x)^2-10^2}{2\cdot 17\cdot(12+x)} = \frac{17^2+x^2-10^2}{2\cdot 17\cdot x}$$

$\Rightarrow 289x + 144x + 24x^2 + x^3 - 100x$

$\Rightarrow 289x + 289 \times 12 + 12x^2 + x^3 - 1200 - 100x$

$\Rightarrow x = 9$

$\therefore$ DE = 9, AC = 12 + 9 = 21

$\therefore$ AC + DF = 30

18. If a, b, c ≥ 4 are integers, not all equal, and 4abc = (a + 3) (b + 3) (c + 3), then what is the value of a + b + c ?

Solution

4abc = (a + 3) (b + 3) (c + 3)

Let a + 3 be divisible by 4 $\Rightarrow$ a = 5

$\Rightarrow$ 20bc = 8 (b + 3) (c + 3)

$\Rightarrow$ 5bc = 2(b + 3) (c + 3)

Let b + 3 be divisible by 5 $\Rightarrow$ b = 7

Then 35c = 20 (c + 3)

7c = 4c + 12

3c = 12 $\Rightarrow$ c = 4

$\therefore$ a + b + c = 16

19. Let N = 6 + 66 + 666 + … + 666 … 6, where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N?

Solution

N = 6 + 66 + 666 + … + 666 … 6 hundred terms

$$\frac{9N}{6} = (10-1)+(10^2-1)+(10^3-1)+...+(10^{100}-1)$$

$$\frac{9N}{6} = 10(1+10+10^2+...+10^{99})-100$$

$$\frac{3N}{2} = \frac{10}{9}(10^{100}-1)-100$$

$$N = \frac{20}{27}(10^{100}-1)-\frac{200}{3}$$

$$= \frac{20}{3}\left[\underbrace{(1111.......1)}_{99\,\text{times}}-10\right]$$

$$N = \frac{20}{3}[111......01]$$

For every 111 – There is one 7.

∴ total 33 times 7 occurs

Now multiply 370370 …. 3670 by 2

We get 740740 ….. 7340

∴ Totally the number of 7's is 33

20. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$.

Solution

The sum of all positive integers 'n' the product of whose digits equal to $n^2 - 15n - 27$

Here $n^2 - 15n - 27$ is an odd number $\forall\ n \in I$

'n' → should be a maximum of two digit number since maximum product of 3 digit number = 729 and minimum value of $n^2 - 15n - 27$ [for 3 digit number]

10000 – 1500 – 27 then it will be greater than 729

For all $n \in \{8, 9, 10, …\}$

The function $n^2 - 15n - 27$ is increasing

When n = 21,

$n^2 - 15n - 27 = (21)^2 - (15)(21) - 27 = 99$

when n = 19,

$n^2 - 15n - 27 = (19)^2 - (15)(19) - 27 = 49$

when n = 17

$n^2 - 15n - 27 = (17)^2 - 15(17) - 27 = 17$

Maximum product of digits of two digit number is 81.

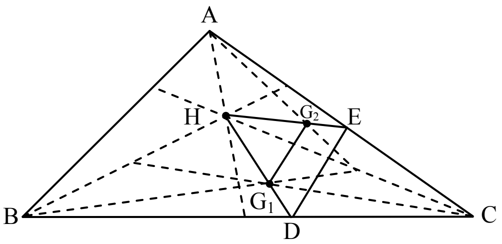
'n' should be less than 21.

∴ $n^2 - 15n - 27$ is negative when 'n' lies between 1 to 15.

Required number n = 17

21. Let ABC be an acute-angled triangle and let H be its orthocentre. Let G_1, G_2 and G_3 be the centroids of the triangles HBC, HCA and HAD, respectively. If the area of triangle $G_1G_2G_3$ is 7 units, what is the area of triangle ABC?

Solution



AB = 2DE … (1)

In ΔHG_1G_2 and ΔHDE

$$\frac{HG_1}{HD} = \frac{G_1G_2}{DE} = \frac{2}{3}$$

$$G_1G_2 = \frac{2}{3}DE = \frac{2}{3}\left(\frac{AB}{2}\right) = \frac{AB}{3}$$

Becomes $\Delta G_1G_2G_3 \sim \Delta ABC$

$$\Rightarrow \frac{\text{Area of }\Delta ABC}{\text{Area of }\Delta G_1G_2G_3} = \frac{(AB)^2}{(G_1G_2)^2} = \left(\frac{AB}{G_1G_2}\right)^2 = \frac{3}{1}$$

$\Rightarrow$ Area of $\Delta ABC = 9 \times$ (Area of $\Delta G_1G_2G_3$)

$\Rightarrow$ Area of $\Delta ABC = 9 \times 7 = 63$

22. A positive integer k is said to be good if there exists a partition of {1, 2, 3, …, 20} in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there?

Solution

Sum of numbers is $\frac{n(n+1)}{2} = \frac{20 \times 21}{2} = 210$

Now $210 = 2 \times 3 \times 5 \times 7$

	Number of partitions	Sum
1	2	105
2	3	70
3	5	42
4	7	30
5	6	35
6	10	21

$\therefore$ k can be any of 21, 30, 35, 47, 70, 105

$\therefore$ Good numbers = 6

If there are two partitions, the partition can be

A = {1, 2, 3, 4, 5, 16, 17, 18, 19, 20}

B = {6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

If the partitions are there, the partitions can be A = {20, 19, 18, 13}, B = {17, 16, 15, 12, 10}, C = {1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14}

If the partitions are 5, they can be

A = {20, 10, 12}, B = {18, 11, 13}, C = {16, 15, 9, 2}, D = {19, 8, 7, 5, 3}

If partitions are 7, they are

A {20, 10}, B = {19, 11}, C = {17, 13}, D = {18, 12}, E = {16, 14}, F = {1, 15, 5}, G = {2, 3, 4, 6, 7, 8}

If the partition are 6, they are

A = {20, 15}, B = {19, 16}, C = {18, 17}, D = {8, 13, 14}, E = {2, 10, 11, 12}, F = {1, 3, 4, 5, 6, 7, 9}

If the partitions are 10, they are

A = {1, 20}, B = {2, 19}, C = {3, 18}, D = {4, 17}, E = {5, 16}, F = {6, 15}, G = {7, 14}, H = {8, 13}, I = {9, 12}, J = {10, 11}

$\therefore$ there are 6 possibilities

23. What is the largest positive integer n such that $\dfrac{a^2}{\frac{b}{29}+\frac{c}{31}}+\dfrac{b^2}{\frac{c}{29}+\frac{a}{31}}+\dfrac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq n(a+b+c)$ holds for all positive real numbers a, b, c?

Solution

$$\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq (a+b+c)n$$

Result: $\dfrac{a^2}{x}+\dfrac{b^2}{y}+\dfrac{c^2}{z} \geq \dfrac{(a+b+c)^2}{x+y+z}$

$$x=\frac{b}{29}+\frac{c}{31},\ y=\frac{c}{29}+\frac{a}{31}, z=\frac{a}{29}+\frac{b}{31}$$

$$\Rightarrow \frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z} \geq \frac{899}{60}(a+b+c)$$

$$n=\left[\frac{899}{60}\right]=14$$

24. If N is the number of triangles of different shapes (i.e., not similar) whose angles are all integers (in degrees), what is $\dfrac{N}{100}$?

Solution

$x + y + z = 180 \qquad (\because$ x, y, z are angles)

$x' + y' + z' = 177$

Total = ${}^{177+2}C_2 \Rightarrow$ Total $=\dfrac{179\times178}{2}=179\times89$

Total = 3!(α β γ) + 3 (ααβ) + ααα = 6(αβγ) + 3(ααβ) + 1

For (ααα), number of ways = 1

For (ααβ), 2α + β = 177, number of ways = (88 cases)

For α, β, γ, number of ways $=\dfrac{179\times89-3\times88-1}{6}$

$$=\frac{15931-265}{6}=2611$$

So the total number of ways, 2611 + 88 + 1 = 2700

N = 2700

$$\Rightarrow \frac{N}{100}=\frac{2700}{100}=27$$

N = 27

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25. Let T be the smallest positive integer which, when divided by 11, 13, 15 leaves remainders in the sets {7, 8, 9}, {1, 2, 3}, {4, 5, 6} respectively. What is the sum of the squares of the digits of T?

Solution

Here given T = {7, 8, 9} (mod 11)

= {1, 2, 3} (mod 13)

= {4, 5, 6} (mod 15)

Since 11, 13 and 15 are pairwise co-prime. Then, by Chinese remainder theorem, there exists a unique solution T modulo (11 × 13 × 15) to the above system of simultaneous congruences.

Let N = 11 × 13 × 15 = 2145 and $m_1 = \frac{11\times13\times15}{11} = 195$

$$m_2 = \frac{11\times13\times15}{13} = 165$$

$$m_3 = \frac{11\times13\times15}{15} = 143$$

Now we have to find

$p_1 = (195)^{-1} \bmod 11 = (8)^{-1} \bmod 11 = 7 \pmod{11}$

$p_2 = (165)^{-1} \bmod 13 = (9)^{-1} \bmod 13 = 3 \pmod{13}$

$p_3 = (143)^{-1} \bmod 15 = (8)^{-1} \bmod 15 = 2 \pmod{15}$

Then $M_1 = m_1p_1 \pmod{2145}$

$= 195 \times 7 \pmod{2145}$

$= 1365 \pmod{2145}$

$M_2 = m_2p_2 \pmod{2145}$

$= 165 \times 3 \pmod{2145}$

$= 495 \pmod{2145}$

$M_3 = 143 \times 2 \pmod{2145}$

$= 286 \pmod{2145}$

So, now T can be given as

$T = 8 \times 1365 + 2 \times 495 + 4 \times 286 \pmod{2145}$

$= 184 \pmod{2145}$

Sum of square of digits = $1^2 + 8^2 + 4^2 = 1 + 64 + 16 = 81$

26. What is the number of ways in which one can choose 60 unit squares from a 11 × 11 chessboard such that no two chosen squares have a side in common?

Solution

Total number of black square in 11 × 11 chessboard = 61

Total number of white square = 60

So, there are two possibilities

Either we can choose 60 black squares from 61 black squares or select all 60 white squares

Which is given by

Total number of ways = ${}^{61}C_{60} + {}^{60}C_{60}$

= 61 + 1 = 62

27. What is the number of ways in which one can colour the squares of a 4×4 chessboard with colours red and blue such that each row as well as each column has exactly two red squares and two blue squares?

Solution

In this, 1^{st} row can be filled by ${}^4C_2$ ways = 6 ways

Then there are three possibilities to fill the rest 3 rows.

Case I: 2^{nd} row is filled same as 1^{st} row

2^{nd}, 3^{rd}, 4^{th} row is filled by one way

So, total ways in case I is ${}^4C_2 \times 1 \times 1 \times 1 = 6$ ways

Case II: Exactly 1 red and 1 blue is interchanged in 2^{nd} row in comparison to 1^{st} row.

So, 2^{nd} row is filled by 2×2 ways

3^{rd} row is filled by 2 way and

4^{th} row is also filled by 1 way

So, total ways in case II is ${}^4C_2 \times 2 \times 2 \times 2 \times 1 = 48$ ways

Case III: Both red and blue is replaced by other in 2^{nd} row as compared to 1^{st} row.

$\Rightarrow 2^{nd}$ row is filled by 1 way

Number of ways to fill 3^{rd} row is ${}^4C_2$

and for 4^{th} row, it is 1

Total number of ways = ${}^4C_2 \times 1 \times 6 \times 1 = 36$

So, from case I, II and III

Total ways = 6 + 48 + 36 = 90

28. Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N.

Solution

$8 \rightarrow (1, 2, 5)$ or $(1, 3, 4)$

$$\text{Number of ways } = \frac{8!}{2!5!1!} \times 3! + \frac{8!}{1!3!4!} \times 3!$$

$$= \left(\frac{8 \times 7 \times 6}{2} + \frac{8 \times 7 \times 6 \times 5}{6}\right) \times 6$$

$$= 56 \times 6\,(3 + 5)$$

$$= 56 \times 48$$

$$= 2688$$

Sum of digits = 24

29. Let D be an interior point of the side BC of a triangle ABC. Let I_1 and I_2 be the incentres of triangles ABD and ACD respectively. Let AI_1 and AI_2 meet BC in E and F respectively. If $\angle BI_1E = 60°$, what is the measure of $\angle CI_2F$ in degrees?

Solution

$\angle BAD = 120° - B$

$\angle CAD = \angle A - (120° - B)$

$= A + B - 120°$

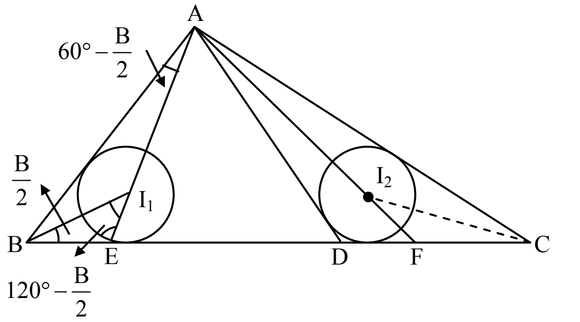
$$\angle FAC = \frac{A + B}{2} - 60° \Rightarrow 90° - \frac{C}{2} - 60°$$

$\Rightarrow 30° - \frac{C}{2}$

$\underline{|AFC} = 180° - \left(C + 30° - \frac{C}{2}\right)$

$= 150° - \frac{C}{2}$

$\underline{|CI_2F} = 180° - \left(150° - \frac{C}{2} + \frac{C}{2}\right) = 30°$



30. Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial in which a_i is a non-negative integer for each $I \in \{0, 1, 2, 3, \ldots, n\}$. If $P(1) = 4$ and $P(5) = 136$, what is the value of $P(3)$?

Solution

$P(1) = a_0 + \ldots + a_n = 4$

$a_0 \geq 0$

$a_n \geq 0$

So atleast four of them must be non-zero possible polynomial $f(x) = x^3 + 2x + 1$ (Intitution)

$f(1) = 1^3 + 2 \times 1 + 1 = 4$

$f(5) = (5)^3 + 2 \times 5 + 1 = 125 + 10 + 1 = 136$

$f(3) = (3)^3 + 2 \times 3 + 1 = 27 + 6 + 1 = 34$

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